Some problems of varying difficulty
SOLUTIONS

1. From the given information we know there are $\frac{3 \cdot 5280 \cdot 60}{100} = 9504$ cars per hour (on average) passing any given point x, south of the overpass.

   Since there is no exit at the overpass, the same number of cars per minute must pass any point y just north of the overpass. Thus $\frac{2 \cdot 5280 \cdot 50}{d} = 9504$ where $d = 55\,\frac{5}{9}$ feet is the average distance per car in each lane north of the overpass.

2. Let n, d and q denote the number of nickels, dimes and quarters respectively. Let k be the total number of coins, and v be their total value in cents. Then

$$n + d + q = k$$

$$5n + 10d + 25q = v$$

are the two equations which Melissa was able to solve, in three unknowns. Each of the equations represents a plane in three dimensions (points $(n, d, q) \in R^3$), and their intersection must be a line L. Clearly that line must have had just one point with integer coordinates in the first octant ($n \geq 0,\ d \geq 0$ *and* $q \geq 0$), for Melissa to solve the problem.

The direction vector for L is obtained as the cross-product of the orthogonal vectors for the two planes: $(1, 1, 1) \times (5, 10, 25) = (15, -20, 5) = 5(3, -4, 1)$. In other words, if (n, d, q) is an integer point on L, the next such point (in either direction) is $(n, d, q) \pm (3, -4, 1)$. Thus, in order for Melissa to have solved the problem with no other information, it must have been true that $d < 4$ and either $n < 3$ or $q < 1$.

But we are given $q \geq 3$ and that $d \geq n + q$ (so $4 > n + q \geq n + 3$, which implies that $n = 0$ and $q = 3$). Then $d = 3$. Thus Tom had three dimes, three quarters, and no nickels, so the value of his change was a dollar and five cents.

3. Think of the 200 yard wide cataract coming *toward* Tom at 3.25 feet per second. Can he stay ahead of it long enough to reach the shore, given that he only swims at 3 fps?

If Tom aims for a point on the edge of the moving water which is d feet (relative to the water) in the direction opposite the cataract, he must travel $\sqrt{d^2 + 300^2}$ feet. At 3 fps, that will take $\frac{1}{3}\sqrt{d^2 + 300^2}$ seconds.

On the other hand, the waterfall requires $\frac{130+d}{3.25}$ seconds to reach that same point. Can Tom arrange that

$$\frac{1}{3}\sqrt{d^2 + 300^2} < \frac{130 + d}{3.25}?$$

By squaring and manipulating, we find this is algebraically equivalent to

$$f(d) = d^2 - 1497.6d + 511056 < 0.$$

Setting the derivative to zero gives the mid-point of the parabola $y = f(d)$ at $d = 748.8$, and $f(748.8) = -49645.44$, which is negative. So the required inequality is satisfiable, and Tom could have escaped (with about one and a half seconds to spare) by aiming for the point on the water edge which was 748.8 feet farther back from the falls. (Unfortunately, he spent so much time solving this problem that he was swept over the waterfall anyway.)

4. The key to such *Nim-type* games is to start at the end of the game and work backwards. If I want to take the last mint, I must force my opponent to leave me with one, four or eight mints at some point in the game. If I leave my opponent with *two* mints, for example, he will be unable to avoid leaving me one mint, so I can win the game. Similarly, if I can leave my opponent with five, he will be unable to avoid leaving me either one or four, allowing me to win the game.

   If we concentrate on numbers, called *targets*, which it is desirable to leave for one's opponent, we find that 0, 2, and 5 are targets. On the other hand, 1, 3, 4, 6 are not targets because anyone left with one of those numbers can immediately convert it to a good leave. Recursively, n is a non-target if at least one of $\{n-1, n-4, n-8\}$ is a target; and n is a target if *all* non-negative elements of $\{n-1, n-4, n-8\}$ are non-targets.

   Continuing to work backward, we find targets $0, 2, 5, 7, 12, 14, 17, 19, 24, 26, 29, \ldots$, with all the intervening numbers being non-targets.

   At this point, we may reasonably guess that the targets are all the non-negative integers which are congruent to zero, two, five or seven modulo twelve.

   If we let $T = \{n \geq 0 \,|\, n \equiv 0, 2, 5, 7 \mod 12\}$, it is an easy exercise to prove that any positive integer outside the set T can be reduced to a member of T by subtracting either 1, 4 or 8 – while no member of T can be so reduced to another member of T.

   Thus the winning player must always leave her opponent with an element of T, and her opponent will be unable to avoid returning a non-target, etc. In our case, beginning with a hundred mints, the first player must remove four mints, changing the pile to 96 (which is congruent to 0 mod 12). Thereafter, whatever her opponent does, she will always be able to change the number back into the set T – until she is able to take the very last mint (since $0 \in T$).

5. Let $S = \sum_{n=1}^{10^{12}} \sqrt{n}$. Let $I = \int_0^{10^{12}} \sqrt{x}\, dx$. Then

$$\begin{aligned} S - I &= \sum_{n=1}^{10^{12}} (\sqrt{n} - \int_{n-1}^{n} \sqrt{x}\, dx) \\ &\leq \sum_{n=1}^{10^{12}} (\sqrt{n} - \sqrt{n-1} = 10^6. \end{aligned}$$

But $I = \frac{2}{3} \cdot 10^{12}\sqrt{10^{12}} = \frac{2}{3} \cdot 10^{18}$. Therefore I is an approximation to S, accurate to more than ten decimal digits.

6. Call an n-digit number *good* if it contains no pairs of adjacent zeros. We may form a good $(n+1)$- digit number by adjoining an appropriate digit to the right of a good n-digit number.

   Let $\beta(n)$ be the number of good n-digit numbers which end with 0 (the right-most digit) and let $\gamma(n)$ be the number of good n-digit numbers which end with something other than zero. Thus $\alpha(n) = \beta(n) + \gamma(n)$.

   Furthermore, $\beta(n+1) = \gamma(n)$ (since a zero can only be appended to the right if the previous right-most digit was non-zero), and $\gamma(n+1) = 9\alpha(n) = 9\beta(n) + 9\gamma(n) = 9\gamma(n-1) + 9\gamma(n)$. Thus $\gamma(n)$ satisfies a second-order linear recurrence with characteristic equation $\lambda^2 - 9\lambda - 9 = 0$, and roots $\lambda = \frac{9 \pm 3\sqrt{13}}{2}$.

   That implies that $\gamma(n)$, hence also $\beta(n)$ and $\alpha(n)$ are of the form

$$A\left(\frac{9+3\sqrt{13}}{2}\right)^n + B\left(\frac{9+3\sqrt{13}}{2}\right)^n.$$

   Thus

$$\lim_{n\to\infty} \frac{\log(\alpha(n))}{n} = \log\left(\frac{9+3\sqrt{13}}{2}\right)$$

   (since the other root has absolute value less than one).

7. Let r denote the length of PQ (and the radius of γ). Then the length of PS equals $\frac{1}{4}r(\sqrt{5}+1)$. Thus the cosine of angle UPS is $\frac{1}{4}(\sqrt{5}+1)$. By computation, angle UPS is approximately 36° ($\frac{\pi}{5}$ radians), but we need to know whether that is the exact answer.

   The sine and cosine of $\theta = \frac{\pi}{5}$ should satisfy

$$(\cos\theta + i\sin\theta)^5 = -1$$

   In particular, the real part (setting $\sin^2\theta = 1 - \cos^2\theta$ requires

$$\cos(\theta)(16\cos^4(\theta) - 20\cos^2(\theta) + 5) = -1,$$

   which is satisfied *exactly* if we substitute $\cos(\theta) = \frac{1}{4}(\sqrt{5}+1)$. Thus angle UPS is exactly $\frac{\pi}{5}$ and angle UPV is *exactly* $\frac{2\pi}{5}$.

8. First, we may try linear and quadratic polynomials for the function $f(x)$. In particular, if $y = ax^2 + bx$ (there is no constant term since $f(0) = 0$), we obtain

$$\int_0^1 (y'^2 + y)dx = \frac{4}{3}\,a^2 + 2ab + b^2 + \frac{1}{3}\,a + \frac{1}{2}\,b.$$

This quadratic function of a and b is minimized when the partial derivatives are zero, i.e. when $a = \frac{1}{4}$ and $b = -\frac{1}{2}$, giving a minimum value of $-\frac{1}{12}$ for the integral, when $y = \frac{1}{4}\,x^2 - \frac{1}{2}\,x$ Now let $y = \frac{1}{4}\,x^2 - \frac{1}{2}\,x + u(x)$ (where $u(0) = 0$). Then

$$\int_0^1 (y'^2 + y)dx = -\frac{1}{12} + \int_0^1 u'^2 dx + \int_0^1 \frac{d}{dx}\,((x-1)u)dx$$

Since the right hand integral is zero and the middle one is non-negative, we've shown that the left integral is always at least $-\frac{1}{12}$, and equality holds only when $y = \frac{1}{4}\,x^2 - \frac{1}{2}\,x$.

9. Let us assume that the sphere is $x^2 + y^2 + z^2 = 1$ so the required planes will pass through the origin.

Let P_1, P_2, P_3 and P_4 be the four points. The only way there can not be a plane through the origin with all four points on one side of it, is if the tetrahedron (the convex hull) of the four points contains the origin. That means that each point (say P_4 for example), is in the negative convex cone of the other three (with apex at the origin).

If three points P_1, P_2 and P_3 are picked at random, the probability that P_4 will be in the convex cone of $-P_1, -P_2$ and $-P_3$, is proportional to the area which that cone subtends on the sphere – the solid angle. Thus we may *average* all such solid angles, formed by three randomly chosen rays from the origin, to find the probability that no effective plane through the origin exists.

Given, three such points, however, there are actually eight solid angles formed (combinations of $\pm P_1$, and $\pm P_2$ and $\pm P_3$), and the average of those eight solid angles is clearly one-eighth of the whole. We may carry out our averaging process over *all* random solid angles by first grouping them into eights (corresponding to triples of lines through the origin). Thus the probability of failure (no plane through the origin exists with all four points on the same side) is one-eighth; and the answer to the problem is seven- eighths.