Some Problems of Varying Difficulty

1. There are three lanes of northbound traffic on the Interrogatory Expressway, until it passes under the Enigma Overpass (where there is no exit). South of the overpass, the average speed of the cars is sixty miles per hour, and in each lane there is a car (on average) every one hundred feet.

   North of the overpass however, there are only two lanes of northbound traffic, and the average speed of the cars is fifty miles per hour. How much closer together are the cars (on average)?

2. John had a pocket full of change. There was no pennies, and nothing larger than a quarter. When he told Melissa how many coins there were altogether, and their total value, she was able to deduce the number of coins of each of the three possible denominations.

   Given that John had at least three quarters, and that at least half of his coins were dimes, how much money did John have in change?

3. Tom, in his inner-tube, is floating lazily down the two-hundred yard wide Muddy River which flows at a speed of 3.25 feet-per-second. He is a hundred yards from either shore when he hears the roar of a cataract a hundred thirty *feet* in front of him.

   Unfortunately, Tom can only swim three feet per second (in still water). Can he escape being carried over the falls? Prove your answer.

4. We've invented a new game for two players. We begin with a large stack of chocolate mint wafers. Each player in turn may eat either one, four or eight wafers. The player who eats the last morsel wins.

   Given that we begin with a hundred wafers, and that neither player makes a mistake, is this game a win for the first player or is it a win for the second player? What strategy should the winning player follow?

5. Compute $\sum_{n=1}^{10^{12}} \sqrt{n}$ to ten-digit accuracy. (Prove that your answer is sufficiently accurate.)

6. By an n-digit decimal number, we mean a string $d_1d_2d_3\ldots d_n$ where each d_i is an integer from zero to nine. Thus 0097 qualifies as a four-digit decimal number.

   Let $\alpha(n)$ denote the number of n-digit decimal numbers which *do not* contain any adjacent zeros. (Thus 0341105 would be counted in $\alpha(7)$, but 0450009 would not.) Prove that

   $$\lim_{n\to\infty} \frac{\log(\alpha(n))}{n}$$

   exists, and compute the limit in closed form.

7. Let PQR be a right angle such that the length of QR is twice the length of PQ. Extend the line PR beyond R to a point T such that the length of RT equals the length of PQ.

   Let S be the point on the line PT which is one-fourth of the way from P to T. Let m be the line through S and perpendicular to the line PR. Let γ be the circle through Q, with center at P. Let U and V be the points where m intersects γ. Find the angle UPV *precisely as a rational multiple of* π (and prove your answer).

8. Let $y = f(x)$ be a continuously function on $[0,1]$, with $f(0) = 0$. Find the largest real number B such that the statement

$$\int_0^1 (y'^2 + y)dx \geq B$$

   is true for all such functions f.

9. Given four random points on a sphere, what is the probability that there exists a plane through the center of the sphere with all four points on the same side of the plane?